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The representation theory of Padé approximants

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Abstract

These notes explore the consequences of simple representation theory of $\mathfrak{sl}_2$ for the expressions for Padé approximants and discuss the role of a generalized Hirota derivative therein.

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1. Introduction

In [2, 3], a theory of Hirota *maps* has been developed in which certain Hirota-like operators are shown to act as intertwining operators between certain tensored representations of $\mathfrak{sl}_2$. This gives a constructive approach to the representation theory which is obviously very simple in this case. Similar operators can be developed for $\mathfrak{sl}_n$ [4]. In addition the superficial resemblance to the classical Hirota derivative is shown to become a precise equivalence in the limit that the representations are allowed to become infinite dimensional. Nevertheless the role of these Hirota *maps* in the theory of integrable systems remains obscure, in particular, the extent to which they allow representation of integrable-like equations and their solutions.

The present paper arose from a study of the structure of Padé approximants and might be seen as a step towards resolving this issue. In particular we consolidate here the definition of Hirota maps in a way that makes very clear the relation between finite and infinite dimensional representations, which introduces a new type of Hirota map satisfying a closed algebra (again $\mathfrak{sl}_2$) and we show how the coefficients of the rational approximation to an analytic function can be interpreted via the representation theory. In part this provides a very efficient way of calculating these coefficients which involves only derivations and not the solutions of large linear systems. Finally we will see how each approximant is associated with a highest weight vector. Relations between these highest weight vectors are of the 'Lozenge' type which occur in the standard theory [5, 7] and in present treatments of integrable discrete systems [6, 8].

The paper is essentially synthetic in its approach, drawing together threads from representation theory, bilinear differential equations and integrable discrete systems.

2. Fundamental modules of the theory

The most basic objects in our approach are the $\mathbb{Z}$ modules $V^{[n]}$ for $n \in \mathbb{Z}$ with basis $\{x^i y^j | i, j \in \mathbb{Z}, i + j = n\}$.

$V^{[n]}$ is an $\mathfrak{sl}_2(\mathbb{Z})$ module under the action

$$\mathbf{e}(x^i y^j) = j x^{i+1} y^{j-1} \tag{2.1}$$

$$\mathbf{f}(x^i y^j) = i x^{i-1} y^{j+1} \tag{2.2}$$

$$\mathbf{h}(x^i y^j) = (i - j) x^i y^j \tag{2.3}$$

where $[\mathbf{e}, \mathbf{f}] = \mathbf{h}$, $[\mathbf{h}, \mathbf{e}] = 2\mathbf{e}$ and $[\mathbf{h}, \mathbf{f}] = -2\mathbf{f}$.

This action is conveniently represented by differential operators,

$$\mathbf{e} = x\partial_y \tag{2.4}$$

$$\mathbf{f} = y\partial_x \tag{2.5}$$

$$\mathbf{h} = x\partial_x - y\partial_y. \tag{2.6}$$

The most important submodules of $V^{[n]}$ are

$$V_0^{[n]} = \mathbb{Z}\langle x^i y^j | i + j = n, i \geqslant 0\rangle \tag{2.7}$$

and

$$V_\infty^{[n]} = \mathbb{Z}\langle x^i y^j | i + j = n, j \geqslant 0\rangle. \tag{2.8}$$

When $n < 0$ $V_0^{[n]}$ and $V_\infty^{[n]}$ have trivial intersection but when $n \geqslant 0$ their intersection

$$V_{\mathrm{glob}}^{[n]} = \mathbb{Z}\langle x^i y^j | i + j = n, i, j \geqslant 0\rangle \tag{2.9}$$

is a finite dimensional $\mathfrak{sl}_2(\mathbb{Z})$-module with basis the $n + 1$ independent, homogeneous monomials of degree n.

There are quotient modules, $V^{[n]} \big/ V_0^{[n]}$ etc and in the case that $n \geqslant 0$,

$$V^{[n]} \big/ V_0^{[n]} \cong V_\infty^{[n]} \big/ V_{\mathrm{glob}}^{[n]}.$$

Note that although $V_{\mathrm{glob}}^{[n]} \lhd V_0^{[n]} \lhd V^{[n]}$, for $n \geqslant 0$, we do not have complete reducibility, e.g. $\mathbf{f}(x^{n+1} y^{-1}) \in V_{\mathrm{glob}}^{[n]}$.

3. Pairing and associated modules

We wish to think of analytic functions as carrying an $\mathfrak{sl}_2(\mathbb{Z})$ action and we do this in the following manner.

Let $F^{[n]}$ be a $\mathbb{Z}$-module with basis $\left\{ f_{i,j}^{[n]} | i + j = n \right\}$. We take as understood an underlying field K and that $f \in F^{[n]}$ is a function $f : \mathbb{Z} \to K$ and we define a pairing $F^{[n]} \times V^{[n]} \to K^{[n]}((x, y))$, formal Laurent series of degree n over $\mathbb{P}^1(K)$ by

$$(f, v) = \sum_{i+j=n} f_{i,j}^{[n]} x^i y^j. \tag{3.1}$$

Then there is an induced action of $\mathfrak{sl}_2(\mathbb{Z})$ on $F^{[n]}$ given by the invariance conditions,

$$(\mathbf{e}(f), v) + (f, \mathbf{e}(v)) = 0 \tag{3.2}$$

$$(\mathbf{f}(f), v) + (f, \mathbf{f}(v)) = 0 \tag{3.3}$$

$$(\mathbf{h}(f), v) + (f, \mathbf{h}(v)) = 0. \tag{3.4}$$

It is easy to see that the action is given explicitly by

$$\mathbf{e}\big(f_{i,j}^{[n]}\big) = -(j+1)f_{i-1,j+1}^{[n]} \tag{3.5}$$

$$\mathbf{f}\big(f_{i,j}^{[n]}\big) = -(i+1)f_{i+1,j-1}^{[n]} \tag{3.6}$$

$$\mathbf{h}\big(f_{i,j}^{[n]}\big) = -(j-i)f_{i,j}^{[n]}. \tag{3.7}$$

The submodule structure of $F^{[n]}$ is not quite the same as for $V^{[n]}$. For instance, there is a finite dimensional submodule only when $n < 0$.

Our initial thrust is to associate sections analytic in x with a quotient module $F_0^{[n]}$ of $F^{[n]}$ and to extend the notion of Hirota-map from $V_{\mathrm{glob}}^{[n]}$ to $F_0^{[n]}$.

4. Analytic quotient modules

Let $\big(V_0^{[n]}\big)^{\perp}$ be the set $\big\{f \in F^{[n]} \,\big|\, (f, v) = 0, \forall\, v \in V_0^{[n]}\big\}$, a submodule of $F_0^{[n]}$. Then we want

$$F_0^{[n]} = F^{[n]} \big/ \big(V_0^{[n]}\big)^{\perp} \tag{4.1}$$

with the $\mathfrak{sl}_2(\mathbb{Z})$ quotient action, so $\mathbf{e}\big(f_{0,n}^{[n]}\big) = 0$.

Then $F_0^{[n]} \times V_0^{[n]} \to K_0^{[n]}((x, y))$, the sections analytic at $x = 0$, i.e.

$$f(x, y) = \sum_{i \geqslant 0, i+j=n} f_{i,j}^{[n]} x^i y^j \tag{4.2}$$

and $\mathbf{e}(f(x, y)) = \mathbf{f}(f(x, y)) = 0$.

5. Algebra of Hirota maps on tensor products of fundamental modules

Hirota maps at their simplest act on 2-fold tensor products of the modules $V^{[n]}$. They induce, by the invariance condition, corresponding maps on tensor products of the $F_0^{[n]}$. Most important is the fact that they commute with the actions of $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{h}$ and so act as intertwining operators for representations.

The maps we will define here extend those defined in [2–4] from $F_{\mathrm{glob}}^{[n]}$ to all of $F^{[n]}$ and also to $V^{[n]}$ which is actually where it is most convenient to start.

Define

$$\mathbb{F} : V^{[n]} \otimes V^{[m]} \to V^{[n+1]} \otimes V^{[m+1]} \tag{5.1}$$

$$\mathbb{E} : V^{[n]} \otimes V^{[m]} \to V^{[n-1]} \otimes V^{[m-1]} \tag{5.2}$$

by

$$\mathbb{F}(a \otimes b) = xa \otimes yb - ya \otimes xb \tag{5.3}$$

$$\mathbb{E}(a \otimes b) = \partial_x a \otimes \partial_y b - \partial_y a \otimes \partial_x b. \tag{5.4}$$

These Hirota maps restrict to tensor products of the $V_0^{[n]}$. It is also easy to show that they satisfy the $\mathfrak{sl}_2$ commutation relations:

$$[\mathbb{E}, \mathbb{F}] = \mathbb{H} \qquad [\mathbb{H}, \mathbb{E}] = 2\mathbb{E} \qquad [\mathbb{H}, \mathbb{F}] = -2\mathbb{F} \tag{5.5}$$

where $\mathbb{H} = \mathbb{I} + \delta$, $\mathbb{I}$ and δ being the unit derivation and degree derivation respectively. Thus $\mathbb{I}$ counts the degree of the tensor product whilst $\delta(a) = na$ for $a \in V^{[n]}$.

Most crucially though $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{H}$ all commute with the $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{h}$ actions and so intertwine modules: if $M \lhd V^{[n]} \otimes V^{[m]}$ then $\mathbb{E}(M) \lhd V^{[n-1]} \otimes V^{[m-1]}$ and $\mathbb{F}(M) \lhd V^{[n+1]} \otimes V^{[m+1]}$.

For instance:

$$\begin{aligned}[\mathbb{F}, \mathbf{e}](1 \otimes 1) &= \mathbb{F}(x\partial_y \otimes 1 + 1 \otimes x\partial_y) - \mathbf{e}(x \otimes y - y \otimes x)\\ &= x^2\partial_y \otimes y - yx\partial_y \otimes x + x \otimes yx\partial_y - y \otimes x^2\partial_y\\ &\quad - x\partial_y x \otimes y - x \otimes x\partial_y y + x\partial_y y \otimes x + y \otimes x\partial_y x\\ &= 0. \end{aligned} \tag{5.6}$$

For higher degree tensor products there are clearly many more operators of this type that can be defined and there are new types too which go to form a complicated algebraic structure which will be the subject of another paper.

6. Hirota maps on tensor products of associated modules

The pairing $F^{[n]}$ and $V^{[n]}$ extends in the obvious way to pairings of tensor products. Thus

$$(f \otimes g, u \otimes v) = (f, u)(g, v) \tag{6.1}$$

and consequently there is an action of the Hirota maps on $F^{[n]} \otimes F^{[m]}$, namely:

$$(\mathbb{E}(f \otimes g), u \otimes v) + (f \otimes g, \mathbb{E}(u \otimes v)) = 0 \tag{6.2}$$

$$(\mathbb{F}(f \otimes g), u \otimes v) + (f \otimes g, \mathbb{F}(u \otimes v)) = 0 \tag{6.3}$$

or, explicitly:

$$\begin{gathered}\mathbb{E} : F^{[n]} \otimes F^{[m]} \to F^{[n+1]} \otimes F^{[m+1]}\\ \mathbb{E}\big(f_{i,j}^{[n]} \otimes f_{k,l}^{[m]}\big) = -(i+1)(l+1) f_{i+1,j}^{[n+1]} \otimes f_{k,l+1}^{[m+1]} + (j+1)(k+1) f_{i,j+1}^{[n+1]} \otimes f_{k+1,l}^{[m+1]}\end{gathered} \tag{6.4}$$

$$\begin{gathered}\mathbb{F} : F^{[n]} \otimes F^{[m]} \to F^{[n-1]} \otimes F^{[m-1]}\\ \mathbb{F}\big(f_{i,j}^{[n]} \otimes f_{k,l}^{[m]}\big) = -f_{i-1,j}^{[n-1]} \otimes f_{k,l-1}^{[m-1]} + f_{i,j-1}^{[n-1]} \otimes f_{k-1,l}^{[m-1]}.\end{gathered} \tag{6.5}$$

These maps intertwine with the $\mathfrak{sl}_2$ action on the tensor products of associated modules. It can also be checked that they restrict to tensor products $F_0^{[n]} \otimes F_0^{[m]}$ of the analytic modules too.

It is not hard to see that the usual plethysm for tensor products of $\mathfrak{sl}_2$ modules generalizes in the following way,

$$F_0^{[n]} \otimes F_0^{[m]} \cong \bigoplus_{p=0}^{\infty} F_0^{[n+m-2p]} \tag{6.6}$$

for $n, m \in \mathbb{Z}$. Further $\mathbb{E}$ is injective and

$$0 \to F_0^{[n-1]} \otimes F_0^{[m-1]} \stackrel{\mathbb{E}}{\hookrightarrow} F_0^{[n]} \otimes F_0^{[m]} \twoheadrightarrow F_0^{[n+m]} \to 0 \tag{6.7}$$

is exact. Similarly,

$$0 \to F_0^{[n+m]} \hookrightarrow F_0^{[n]} \otimes F_0^{[m]} \stackrel{\mathbb{F}}{\twoheadrightarrow} F_0^{[n-1]} \otimes F_0^{[m-1]} \to 0 \tag{6.8}$$

is exact also.

7. Casimirs

The Casimir operator

$$C = \mathbf{ef} + \mathbf{fe} + \tfrac{1}{2}\mathbf{h}^2 \tag{7.1}$$

commutes with the Hirota maps and takes the value $\frac{1}{2}\delta(\delta+2)$ on the $V^{[n]}$ where $\delta = x\partial_x + y\partial_y$ is the degree derivation, $\delta(x^i y^j) = i + j$. It will enable us to calculate n for any representation isomorphic to some $F_0^{[n]}$.

8. Padé approximants

A comprehensive introduction to the structural theory of Padé approximants is given in [9]. We here summarize the notation we shall use for rational approximations of analytic functions and explain the link with representation theory. We will provide some detailed tables of coefficients in an appendix. These can be used to reinforce later statements empirically.

Given a function analytic in z we seek to approximate it to order z^{n+m+1} as a ratio of polynomials in z, the numerator of degree n and the denominator of degree m. From our point of view it is convenient to homogenize the functions involved by putting $z = x/y$. We use the following notation:

$$P^{[n,m]}(x, y) = p_{0,n}^{[n,m]} y^n + p_{1,n-1}^{[n,m]} x y^{n-1} + \cdots + p_{n,0}^{[n,m]} x^n \tag{8.1}$$

$$Q^{[n,m]}(x, y) = q_{0,m}^{[n,m]} y^m + q_{1,m-1}^{[n,m]} x y^{m-1} + \cdots + q_{m,0}^{[n,m]} x^m \tag{8.2}$$

$$F^{[n,m]} = \sum_{i=0}^{\infty} f_{i,\sigma-i}^{[n,m]} x^i y^{\sigma-i} \tag{8.3}$$

where $\sigma = n - m$,

$$\frac{P^{[n,m]}(x, y)}{Q^{[n,m]}(x, y)} = F^{[n,m]} \sim y^{\sigma} f(z) \bmod z^{n+m+1} \tag{8.4}$$

and $f(z)$ is the given function that is to be approximated

$$f(z) = \sum_{i=0}^{\infty} f_i z^i. \tag{8.5}$$

It is the coefficients $f_0, f_1, \ldots, f_{n+m}$ which appear in the well-known expressions for the coefficients of $P^{[n,m]}$ and $Q^{[n,m]}$, but, in the present treatment we have to be aware that, being the first $n+m+1$ coefficients $f_{i,\sigma-i}^{[n,m]}$ of $F^{[n,m]}$ they are to be interpreted carefully as dependent on n and m.

Multiplying through by $Q^{[n,m]}(x, y)$ in (8.4) and equating coefficients of monomials $x^i y^j$ gives an infinite set of linear equations for the coefficients of $P^{[n,m]}$ and $Q^{[n,m]}$ in terms of the $n+m+1$ 'fundamental variables' $f_{0,\sigma}^{[n,m]}, f_{1,\sigma-1}^{[n,m]}, \ldots, f_{n+m,-2m}^{[n,m]}$. The coefficients of $P^{[n,m]}$ and $Q^{[n,m]}$ are, of course, determined only up to some overall factor. Further the remaining coefficients $f_{i,\sigma-i}^{[n,m]}$ for $i > n+m$ are also determined by these fundamental ones.

From the representation theoretic viewpoint which we are developing the polynomials $P^{[n,m]}$ and $Q^{[n,m]}$ correspond to finite dimensional representations of $\mathfrak{sl}_2$ and $F^{[n,m]}$ to an infinite dimensional representation. Relation (8.4) between $P^{[n,m]}$, $Q^{[n,m]}$ and $F^{[n,m]}$ is invariant under the action of **e**, **f** and **h**. A simple consequence of this is that once we have identified the nature of the representation carried by $F^{[n,m]}$ and expressed, say, the single coefficient $Q_{0,m}^{[n,m]}$ in terms of the fundamental variables, all the other coefficients follow by applications of **f**: that is, we can use the representation theory to construct the solution to the infinite linear system for the coefficients without having to invert (for large m and n) large matrices.

We shall answer the following questions:

1. How can we construct $F^{[n,m]}$ from the module $F^{[\sigma]}$, $\sigma = n-m$, that we introduced earlier on?
2. How do we construct $q^{[n,m]}_{0,m}$ from $F^{[n,m]}$?
3. What recurrence relations hold between the $q^{[n,m]}_{0,m}$ as n and m vary?
4. How do we construct $q^{[n,m]}_{0,m}$ using the Hirota maps?

9. From $F^{[n-m]}$ to $F^{[n,m]}$

Consider the case that $m > n$. This is actually sufficient to cover the general case since for $m \leqslant n$ the rational function simplifies to the sum of a polynomial and a rational function with $m > n$. This is simply an analytic expression of the fact that for $\sigma \geqslant 0$, $V_0^{[\sigma]}$ has a finite dimensional submodule. Nevertheless the formulae we will develop also hold when $n \geqslant m$.

$F^{[n,m]}(x, y)$ is clearly associated with what we have called an analytic module of the kind $F_0^{[\sigma]}$ for $\sigma = n - m$ but with additional relations.

The coefficients of $Q^{[n,m]}$ satisfy the linear system,

$$\begin{pmatrix} f^{[\sigma]}_{n+1} & f^{[\sigma]}_{n} & f^{[\sigma]}_{n-1} & \dots & f^{[\sigma]}_{0} & 0 & \dots \\ f^{[\sigma]}_{n+2} & f^{[\sigma]}_{n+1} & f^{[\sigma]}_{n} & \dots & f^{[\sigma]}_{1} & f^{[\sigma]}_{0} & \dots \\ \vdots & & & & & & 0 \\ f^{[\sigma]}_{m} & f^{[\sigma]}_{m-1} & & & & & f^{[\sigma]}_{0} \\ \vdots & & & & & & \vdots \\ f^{[\sigma]}_{n+m+1} & f^{[\sigma]}_{n+m} & f^{[\sigma]}_{n+m-1} & & & & f^{[\sigma]}_{n+1} \end{pmatrix} \begin{pmatrix} q^{[n,m]}_{0,m} \\ q^{[n,m]}_{1,m-1} \\ \\ \vdots \\ \\ q^{[n,m]}_{m,0} \end{pmatrix} = 0 \tag{9.1}$$

where we have abbreviated the suffices of the f in an obvious way to simplify the presentation.

Hence the $(m + 1) \times (m + 1)$ determinant of coefficients of $f^{[\sigma]}$ must vanish. Call this determinant $\delta^{[n,m]}$. It is easy to see that it is of order $m + 1$ in $f^{[\sigma]}_{n+1}$ and that

$$\delta^{[n,m]} = \left(f^{[\sigma]}_{n+1}\right)^{m+1} - m f^{[\sigma]}_{n+2} f^{[\sigma]}_{n} \left(f^{[\sigma]}_{n+1}\right)^{m-1} + \cdots. \tag{9.2}$$

The condition $\delta^{[n,m]} = 0$ is to be regarded as a relation which defines f_{n+m+1} in terms of the fundamental variables. As there can be no relation between the fundamental variables, it must be that

$$\mathbf{e}(\delta^{[n,m]}) = 0.$$

Consequently, $\delta^{[n,m]} \in \mathrm{Symm}\left(\bigotimes^{m+1} F_0^{[\sigma]}\right)$ and is a highest weight vector.

Relations defining the further coefficients of $f^{[\sigma]}$ will follow by applying the operator $\mathbf{f}$ to $\delta^{[n,m]}$.

Application of the Casimir to $\delta^{[n,m]}$ gives

$$C(\delta^{[n,m]}) = \tfrac{1}{2}(m + 1)(n + m + 2)((m + 1)(n + m + 2) - 2)\delta^{[n,m]}.$$

This careful calculation requires knowing only the two terms shown in (9.2) which are mixed under the application of C.

Thus we can identify $\delta^{[n,m]}$ as the highest weight vector of a module isomorphic to $F_0^{[-(m+1)(m+n+2)]}$ inside $\mathrm{Symm}\left(\bigotimes^{m+1} F_0^{[\sigma]}\right)$.

We therefore arrive at the following construction for the module corresponding to the function $F^{[n,m]}$. Take the symmetric $m + 1$ fold tensor product of $F^{[\sigma]}$ and factor out the submodule isomorphic to $F_0^{[-(m+1)(m+n+2)]}$ generated by $\delta^{[n,m]}$. Then the $f^{[n,m]}_{i,n-m-i}$ are generated by the quotient action of powers of $\mathbf{f}$ acting on $f^{[\sigma]}_{0,\sigma}$, $\sigma = n - m$.

10. From $F^{[n,m]}$ to $q_{0,m}^{[n,m]}$

As the coefficients of $Q^{[n,m]}$ form a zero eigenvector of the matrix of f-coefficients in (9.1) they must themselves be, up to a common factor, the signed cofactors of the determinant $\delta^{[n,m]}$. Thus

$$q_{0,m}^{[n,m]} = \lambda \begin{vmatrix} f_{n+1}^{[\sigma]} & f_n^{[\sigma]} & \cdots & \\ f_{n+2}^{[\sigma]} & f_{n+1}^{[\sigma]} & \cdots & \\ \vdots & & & \vdots \\ f_{n+m}^{[\sigma]} & f_{n+m-1}^{[\sigma]} & & f_{n+1}^{[\sigma]} \end{vmatrix} \tag{10.1}$$

$$q_{1,m-1}^{[n,m]} = -\lambda \begin{vmatrix} f_{n+2}^{[\sigma]} & f_n^{[\sigma]} & \cdots & \\ f_{n+3}^{[\sigma]} & f_{n+1}^{[\sigma]} & \cdots & \\ \vdots & & & \vdots \\ f_{n+m+1}^{[\sigma]} & f_{n+m-1}^{[\sigma]} & & f_{n+1}^{[\sigma]} \end{vmatrix} \tag{10.2}$$

etc but we have to choose λ to ensure that the relation

$$\mathbf{f}\big(q_{0,m}^{[n,m]}\big) = -q_{1,m-1}^{[n,m]}$$

holds, so that the $q_{i,m-i}^{[n,m]}$ do indeed form a finite dimensional representation.

It is easy to see that the above can be written as

$$q_{0,m}^{[n,m]} = \lambda\delta^{[n,m-1]} \tag{10.3}$$

$$q_{1,m-1}^{[n,m]} = \frac{\lambda}{n+m+1} f\,(\delta^{[n,m-1]}) \tag{10.4}$$

which implies that

$$\lambda = (\delta^{[n,m-1]})^{-\frac{n+m+2}{n+m+1}}$$

and, in consequence,

$$q_{0,m}^{[n,m]} = (\delta^{[n,m-1]})^{-\frac{1}{n+m+1}} \tag{10.5}$$

$$p_{0,n}^{[n,m]} = f_0 q_{0,m}^{[n,m]}. \tag{10.6}$$

11. Examples

In order to put some intuitive meat around this rather minimalist skeleton we here derive some Padé approximant formula as described above. We have implemented the representation theory in MAPLE because the manipulations quickly become a headache otherwise. These formulae should be compared with those which are simply obtained using the MAPLE Padé command. The argument above works for $n \geqslant 0$, so we illustrate these cases also.

11.1. $n = 0, m = 1$

$$\delta^{[0,1]} = \begin{vmatrix} f_1 & f_0 \\ f_2 & f_1 \end{vmatrix}$$

$$q_0^{[0,1]} = f_1^{-\frac{1}{2}}$$

$$
\begin{aligned}
q_1^{[0,1]} &= -\mathbf{f}\big(f_1^{-\frac{1}{2}}\big) = -f_1^{-\frac{3}{2}} f_2 = -f_0^{-1} f_1^{\frac{1}{2}} \quad \bmod \delta^{[0,1]} \\
q_2^{[0,1]} &= -\frac{1}{2}\mathbf{f}\big(-f_0^{-1} f_1^{\frac{1}{2}}\big) = 0 \quad \bmod \delta^{[0,1]} \\
p_0^{[0,1]} &= f_0 f_1^{-\frac{1}{2}} \\
p_1^{[0,1]} &= -\mathbf{f}\big(f_0 f_1^{-\frac{1}{2}}\big) = 0 \quad \bmod \delta^{[0,1]} \\
f^{[0,1]}(x, y) &= \frac{f_0 f_1^{-\frac{1}{2}}}{y f_1^{-\frac{1}{2}} - x f_0^{-1} f_1^{\frac{1}{2}}} = \frac{f_0^2}{y f_0 - x f_1}.
\end{aligned} \tag{11.1}
$$

11.2. $n = 1, m = 1$

$$
\begin{aligned}
\delta^{[1,1]} &= \begin{vmatrix} f_2 & f_1 \\ f_3 & f_2 \end{vmatrix} \\
q_0^{[1,1]} &= f_2^{-\frac{1}{3}} \\
q_1^{[1,1]} &= -\mathbf{f}\big(f_2^{-\frac{1}{3}}\big) = -f_2^{-\frac{4}{3}} f_3 = -f_1^{-1} f_2^{\frac{2}{3}} \quad \bmod \delta^{[1,1]} \\
q_2^{[1,1]} &= -\frac{1}{2}\mathbf{f}\big(-f_1^{-1} f_2^{\frac{2}{3}}\big) = 0 \quad \bmod \delta^{[1,1]} \\
p_0^{[1,1]} &= f_0 f_2^{-\frac{1}{3}} \\
p_1^{[1,1]} &= -\mathbf{f}\big(f_0 f_2^{-\frac{1}{3}}\big) = -f_1^{-1} f_2^{-\frac{1}{3}}\big(f_0 f_2 - f_1^2\big) \quad \bmod \delta^{[1,1]} \\
p_2^{[1,1]} &= -\frac{1}{2}\mathbf{f}\big(f_1^{-1} f_2^{-\frac{1}{3}}\big(f_0 f_2 - f_1^2\big)\big) = 0 \quad \bmod \delta^{[1,1]} \\
f^{[1,1]}(x, y) &= \frac{y f_0 f_2^{-\frac{1}{3}} - x f_1^{-1} f_2^{-\frac{1}{3}}\big(f_0 f_2 - f_1^2\big)}{y f_2^{-\frac{1}{3}} - x f_1^{-1} f_2^{\frac{2}{3}}} \\
&= \frac{y f_0 f_1 - x\big(f_0 f_2 - f_1^2\big)}{y f_1 - x f_2}.
\end{aligned} \tag{11.2}
$$

11.3. $n = 0, m = 2$

$$
\begin{aligned}
\delta^{[0,2]} &= \begin{vmatrix} f_1 & f_0 & 0 \\ f_2 & f_1 & f_0 \\ f_3 & f_2 & f_1 \end{vmatrix} \\
q_0^{[0,2]} &= \big(f_1^2 - f_0 f_2\big)^{-\frac{1}{3}} \\
q_1^{[0,2]} &= -\mathbf{f}\big(\big(f_1^2 - f_0 f_2\big)^{-\frac{1}{3}}\big) = \big(f_1^2 - f_0 f_2\big)^{-\frac{2}{3}}(f_0 f_3 - f_1 f_2) \\
&= -f_0^{-1} f_1 \big(f_1^2 - f_0 f_2\big)^{-\frac{1}{3}} \quad \bmod \delta^{[0,2]} \\
q_2^{[0,2]} &= -\frac{1}{2}\mathbf{f}\big(-f_0^{-1} f_1 \big(f_1^2 - f_0 f_2\big)^{-\frac{1}{3}}\big) \\
&= -f_0^{-2}\big(f_1^2 - f_0 f_2\big)^{\frac{2}{3}} \quad \bmod \delta^{[0,2]} \\
q_3^{[0,2]} &= -\frac{1}{3}\mathbf{f}\big(-f_0^{-2}\big(f_1^2 - f_0 f_2\big)^{\frac{2}{3}}\big) = 0 \quad \bmod \delta^{[0,2]}
\end{aligned}
$$

$$
\begin{aligned}
p_0^{[0,2]} &= f_0\big(f_1^2 - f_0 f_2\big)^{-\frac{1}{3}} \\
p_1^{[0,2]} &= -\mathbf{f}\big(f_0\big(f_1^2 - f_0 f_2\big)^{-\frac{1}{3}}\big) = 0 \quad \mod \delta^{[0,2]} \\
f^{[0,2]}(x, y) &= \frac{f_0\big(f_1^2 - f_0 f_2\big)^{-\frac{1}{3}}}{y^2\big(f_1^2 - f_0 f_2\big)^{-\frac{1}{3}} - yxf_0^{-1} f_1\big(f_1^2 - f_0 f_2\big)^{-\frac{1}{3}} + x^2 f_0^{-2}\big(f_1^2 - f_0 f_2\big)^{\frac{2}{3}}} \\
&= \frac{f_0^3}{y^2 f_0^2 - xyf_0 f_1 + x^2\big(f_1^2 - f_0 f_2\big)}.
\end{aligned}
\tag{11.3}
$$

11.4. $n = 2, m = 1$

$$
\begin{aligned}
\delta^{[2,1]} &= \begin{vmatrix} f_3 & f_2 \\ f_4 & f_3 \end{vmatrix} \\
q_0^{[2,1]} &= f_3^{-\frac{1}{4}} \\
q_1^{[2,1]} &= -\mathbf{f}\big(f_3^{-\frac{1}{4}}\big) = -f_2^{-1} f_3^{\frac{3}{4}} \quad \mod \delta^{[2,1]} \\
q_2^{[2,1]} &= -\frac{1}{2}\mathbf{f}\big(-f_2^{-1} f_3^{\frac{3}{4}}\big) = 0 \quad \mod \delta^{[2,1]} \\
p_0^{[2,1]} &= f_0 f_3^{-\frac{1}{4}} \\
p_1^{[2,1]} &= -\mathbf{f}\big(f_0 f_3^{-\frac{1}{4}}\big) = -f_2^{-1} f_3^{-\frac{1}{4}}(f_0 f_3 - f_1 f_2) \quad \mod \delta^{[2,1]} \\
p_2^{[2,1]} &= -\frac{1}{2}\mathbf{f}\big(-f_2^{-1} f_3^{-\frac{1}{4}}(f_0 f_3 - f_1 f_2)\big) \\
&= -f_3^{-\frac{1}{4}} f_2^{-1}\big(f_3 f_1 - f_2^2\big) \quad \mod \delta^{[2,1]} \\
p_3^{[2,1]} &= -\frac{1}{3}\mathbf{f}\big(-f_3^{-\frac{1}{4}} f_2^{-1}\big(f_3 f_1 - f_2^2\big)\big) = 0 \quad \mod \delta^{[2,1]} \\
f^{[2,1]}(x, y) &= \frac{y^2 f_0 f_3^{-\frac{1}{4}} - xyf_2^{-1} f_3^{-\frac{1}{4}}(f_0 f_3 - f_1 f_2) - x^2 f_3^{-\frac{1}{4}} f_2^{-1}\big(f_3 f_1 - f_2^2\big)}{yf_3^{-\frac{1}{4}} - xf_2^{-1} f_3^{\frac{3}{4}}} \\
&= \frac{y^2 f_0 f_2 - xy(f_0 f_3 - f_1 f_2) - x^2\big(f_3 f_1 - f_2^2\big)}{yf_2 - xf_3}.
\end{aligned}
\tag{11.4}
$$

11.5. $n = 1, m = 2$

$$
\begin{aligned}
\delta^{[1,2]} &= \begin{vmatrix} f_2 & f_1 & f_0 \\ f_3 & f_2 & f_1 \\ f_4 & f_3 & f_2 \end{vmatrix} \\
q_0^{[1,2]} &= \big(f_2^2 - f_3 f_1\big)^{-\frac{1}{4}} \\
q_1^{[1,2]} &= -\mathbf{f}\big(\big(f_2^2 - f_3 f_1\big)^{-\frac{1}{4}}\big) \\
&= -\big(f_2^2 - f_3 f_1\big)^{-1/4}\big(f_1^2 - f_0 f_2\big)^{-1}(f_1 f_2 - f_3 f_0) \quad \mod \delta^{[1,2]} \\
q_2^{[1,2]} &= -\frac{1}{2}\mathbf{f}\big(-\big(f_2^2 - f_3 f_1\big)^{-1/4}\big(f_1^2 - f_0 f_2\big)^{-1}(f_1 f_2 - f_3 f_0)\big) \\
&= \big(f_1^2 - f_0 f_2\big)^{-1}\big(f_2^2 - f_3 f_1\big)^{3/4} \quad \mod \delta^{[1,2]}
\end{aligned}
$$

$$
\begin{aligned}
q_3^{[1,2]} &= -\frac{1}{3}\mathbf{f}\big(\big(f_1^2 - f_0 f_2\big)^{-1}\big(f_2^2 - f_3 f_1\big)^{3/4}\big) = 0 \mod \delta^{[1,2]}\\
p_0^{[1,2]} &= f_0\big(f_2^2 - f_1 f_3\big)^{-\frac{1}{4}}\\
p_1^{[1,2]} &= -\mathbf{f}\big(f_0\big(f_2^2 - f_1 f_3\big)^{-\frac{1}{4}}\big)\\
&= \big(f_1^2 - f_0 f_2\big)^{-1}\big(f_2^2 - f_1 f_3\big)^{-1/4}\big(f_1^3 - 2 f_0 f_1 f_2 + f_0^2 f_3\big) \mod \delta^{[1,2]}\\
p_2^{[1,2]} &= -\frac{1}{2}\mathbf{f}\big(\big(f_1^2 - f_0 f_2\big)^{-1}\big(f_2^2 - f_1 f_3\big)^{-1/4}\big(f_1^3 - 2 f_0 f_1 f_2 + f_0^2 f_3\big)\big)\\
&= 0 \mod \delta^{[1,2]}\\
f^{[1,2]}(x,y) &= \frac{y f_0\big(f_1^2 - f_0 f_2\big) + x\big(f_1^3 - 2 f_0 f_1 f_2 + f_0^2 f_3\big)}{y^2\big(f_1^2 - f_0 f_2\big) - xy(f_1 f_2 - f_0 f_3) + x^2\big(f_2^2 - f_1 f_3\big)}.
\end{aligned} \tag{11.5}
$$

11.6. $n = 0, m = 3$

$$
\begin{aligned}
\delta^{[0,3]} &= \begin{vmatrix} f_1 & f_0 & 0 & 0 \\ f_2 & f_1 & f_0 & 0 \\ f_3 & f_2 & f_1 & f_0 \\ f_4 & f_3 & f_2 & f_1 \end{vmatrix}\\
q_0^{[0,3]} &= \big(f_1^3 - 2 f_0 f_1 f_2 + f_0^2 f_3\big)^{-\frac{1}{4}}\\
q_1^{[0,3]} &= -\mathbf{f}\big(\big(f_1^3 - 2 f_0 f_1 f_2 + f_0^2 f_3\big)^{-\frac{1}{4}}\big)\\
&= -f_0^{-1}\big(f_1^3 - 2 f_0 f_1 f_2 + f_0^2 f_3\big)^{-\frac{1}{4}} f_1 \mod \delta^{[0,3]}\\
q_2^{[0,3]} &= -\frac{1}{2}\mathbf{f}\big(-f_0^{-1}\big(f_1^3 - 2 f_0 f_1 f_2 + f_0^2 f_3\big)^{-\frac{1}{4}} f_1\big)\\
&= f_0^{-2}\big(f_1^3 - 2 f_0 f_1 f_2 + f_0^2 f_3\big)^{-\frac{1}{4}}\big(f_1^2 - f_0 f_2\big) \mod \delta^{[0,3]}\\
q_3^{[0,3]} &= -\frac{1}{3}\mathbf{f}\big(f_0^{-2}\big(f_1^3 - 2 f_0 f_1 f_2 + f_0^2 f_3\big)^{-\frac{1}{4}}\big(f_1^2 - f_0 f_2\big)\big)\\
&= -f_0^{-3}\big(f_1^3 + f_0^2 f_3 - 2 f_0 f_1 f_2\big)^{3/4} \mod \delta^{[0,3]}\\
q_4^{[0,3]} &= -\frac{1}{4}\mathbf{f}\big(-f_0^{-3}\big(f_1^3 + f_0^2 f_3 - 2 f_0 f_1 f_2\big)^{3/4}\big) = 0 \mod \delta^{[0,3]}\\
p_0^{[0,3]} &= f_0\big(f_1^3 - 2 f_0 f_1 f_2 + f_0^2 f_3\big)^{-\frac{1}{4}}\\
p_1^{[0,3]} &= -\mathbf{f}\big(f_0\big(f_1^3 - 2 f_0 f_1 f_2 + f_0^2 f_3\big)^{-\frac{1}{4}}\big) = 0 \mod \delta^{[0,3]}\\
f^{[0,3]}(x,y) &= \frac{f_0^4}{y^3 f_0^3 - y^2 x f_0^2 f_1 + y x^2 f_0\big(f_1^2 - f_0 f_2\big) - x^3\big(f_1^3 + f_0^2 f_3 - 2 f_0 f_1 f_2\big)}.
\end{aligned} \tag{11.6}
$$

12. Recurrence relations for highest weight vectors

There is a family of very simple recurrence relation between the $\delta^{[n,m]}$ which arise from a classical determinantal identity (see p 99 of [1]). Suppose that A is an $(n+2)\times(n+2)$ matrix with coefficients a_{ij} and let ${}_iA_j$ denote the matrix with the ith row and jth column removed. Then

$$
|A||{}_{i,k}A_{l,j}| = |{}_iA_j||{}_kA_l| - |{}_iA_l||{}_kA_j|. \tag{12.1}
$$

Then take $A = \delta^{[n+1,m+2]}$, $i = n$, $k = 1$, $l = 1$ and $j = 2$. We obtain

$$\delta^{[n+1,m+2]}\delta^{[n,m]} = \frac{1}{n+m+3} f(\delta^{[n,m+1]})\delta^{[n+1,m+1]} - \frac{1}{n+m+4} f(\delta^{[n+1,m+1]})\delta^{[n,m+1]}. \tag{12.2}$$

This is a relation of *Lozenge* type [7]. It is also a characteristic type of relation within discrete integrable systems theory [6, 8].

13. The role of Hirota maps

All the $\delta^{[n,1]}$ can be constructed as multiple applications of the Hirota maps. It is straightforward to establish, by induction, the following formula:

$$\frac{1}{n!}\mathbb{E}^n\big(f^{[\sigma]}_{0,\sigma} \otimes f^{[\sigma]}_{0,\sigma}\big) = \sum_{p=0}^{n} (-1)^p C^\sigma_{n,p} f^{[\sigma+n]}_{n-p,\sigma+p} \otimes f^{[\sigma+n]}_{p,\sigma+n-p} \tag{13.1}$$

where the integers $C^\sigma_{n,p}$ are given by

$$C^\sigma_{n,p} = (\sigma+1)(\sigma+2)\cdots(\sigma+p)(\sigma+1)(\sigma+2)\cdots(\sigma+n-p). \tag{13.2}$$

If n is even and we take $2\sigma + n + 4 = 0$ then all but three of the $C^\sigma_{n,p}$ vanish and the non-vanishing ones are

$$\begin{aligned}
C^\sigma_{n,n/2-1} &= (\sigma+1)^2(\sigma+2)^2\cdots(\sigma+n/2-1)^2(\sigma+n/2)(\sigma+n/2+1)\\
C^\sigma_{n,n/2} &= (\sigma+1)^2(\sigma+2)^2\cdots(\sigma+n/2-1)^2(\sigma+n/2)^2\\
C^\sigma_{n,n/2+1} &= (\sigma+1)^2(\sigma+2)^2\cdots(\sigma+n/2-1)^2(\sigma+n/2)(\sigma+n/2+1).
\end{aligned} \tag{13.3}$$

Hence, for $\sigma < -2$ choose $n = -2\sigma - 4$ and we have

$$\begin{aligned}
\frac{1}{n!}\mathbb{E}^n\big(f^{[-n/2-2]}_{0,-n/2-2} \otimes f^{[-n/2-2]}_{0,-n/2-2}\big) &= (n/2+1)^2(n/2)^2(n/2-1)^2\cdots 2^2\\
&\quad\times\left(\frac{1}{2} f^{[n/2-2]}_{n/2+1,-3} \otimes f^{[n/2-2]}_{n/2-1,-1} - f^{[n/2-2]}_{n/2,-2} \otimes f^{[n/2-2]}_{n/2,-2} + \frac{1}{2} f^{[n/2-2]}_{n/2-1,-1} \otimes f^{[n/2-2]}_{n/2+1,-3}\right).
\end{aligned} \tag{13.4}$$

Replacing n with $2n$ and symmetrizing over the tensor products to obtain polynomials we obtain

$$\begin{aligned}
&\frac{1}{(2n)!}\operatorname{Symm}\big(\mathbb{E}^{2n}\big(f^{[-n-2]}_{0,-n-2} \otimes f^{[-n-2]}_{0,-n-2}\big)\big)\\
&\qquad = -(n+1)!^2\big(\big(f^{[n-2]}_{n,-2}\big)^2 - f^{[n-2]}_{n-1,-1} f^{[n-2]}_{n+1,-3}\big)\\
&\qquad = -(n+1)!^2\delta^{[n-1,1]}.
\end{aligned} \tag{13.5}$$

By introducing suitable generalizations of $\mathbb{E}$ appropriate to higher degree tensor products formulae for the other $\delta^{[n,m]}$ will be obtained but this would necessitate extending the current paper considerably and the details will be left to a further publication.

14. Discussion

We can summarize the results of this paper thus:

- We have introduced some new Hirota maps modelled on but significantly extending both the classical Hirota derivative and the maps introduced by the author in previous publications.

- We have shown that the theory of Padé approximants has a natural interpretation in terms of certain infinite dimensional representations of $\mathfrak{sl}_2$.
- We have used the representation theory to express general formulae for the form of the coefficients in Padé approximants in terms of the action of the algebra on a set of simple coefficients. General expressions for the approximants are obtained by iteration of simple differential operators. This obviates the need to invert large systems of linear equations.
- We have defined this family of coefficients in terms of a family of polynomial functions, $\delta^{[n,m]}$, which are interpreted as highest weight vectors in a tensor product of degree $m+1$. The $\delta^{[n,m]}$ also generate submodules of relations on the form of the analytic expansion of the approximant.
- We have shown that the $\delta^{[n,m]}$ satisfy four-point (Lozenge-type) relations.
- Finally, for the simplest (quadratic) $\delta^{[n,1]}$, we have shown explicitly how they arise under applications of the simplest Hirota maps.

In further work we will discuss the full Hirota operator algebra on arbitrary tensor products and the derivation of the $\delta^{[n,m]}$.

A simple consequence of the above is that every rational function satisfies a multilinear differential equation. For the coefficients $f^{[n,m]}_{i,n-m-i}$ of the analytic expansion of the $[n,m]$ Padé approximant are, up to a factor of $i!$, the derivatives evaluated at $z = 0$. But then the relation $\delta^{[n,m]} = 0$ is a differential equation of degree $m+1$ and order $n+m+1$ satisfied by f. Conversely, given the order of this differential equation, its general solution is precisely the general rational function (up to an overall constant factor) with numerator and denominator of degrees n and m respectively. Further, the fact that this differential equation is in Hirota form (as has been claimed above but actually shown only when $m = 1$) must be of relevance to the theory of integrable systems.

In addition, the lozenge or four-point relation is a staple of treatments of integrable discrete geometry, so its appearance here in tandem with the Hirota derivative, a denizen of smooth integrable systems theory, both wearing representation theoretic clothes is of considerable interest.

This synthesis of representation theory, integrable systems, approximation theory and discrete geometry appears to be very fruitful and is certainly beautiful if not deep.

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